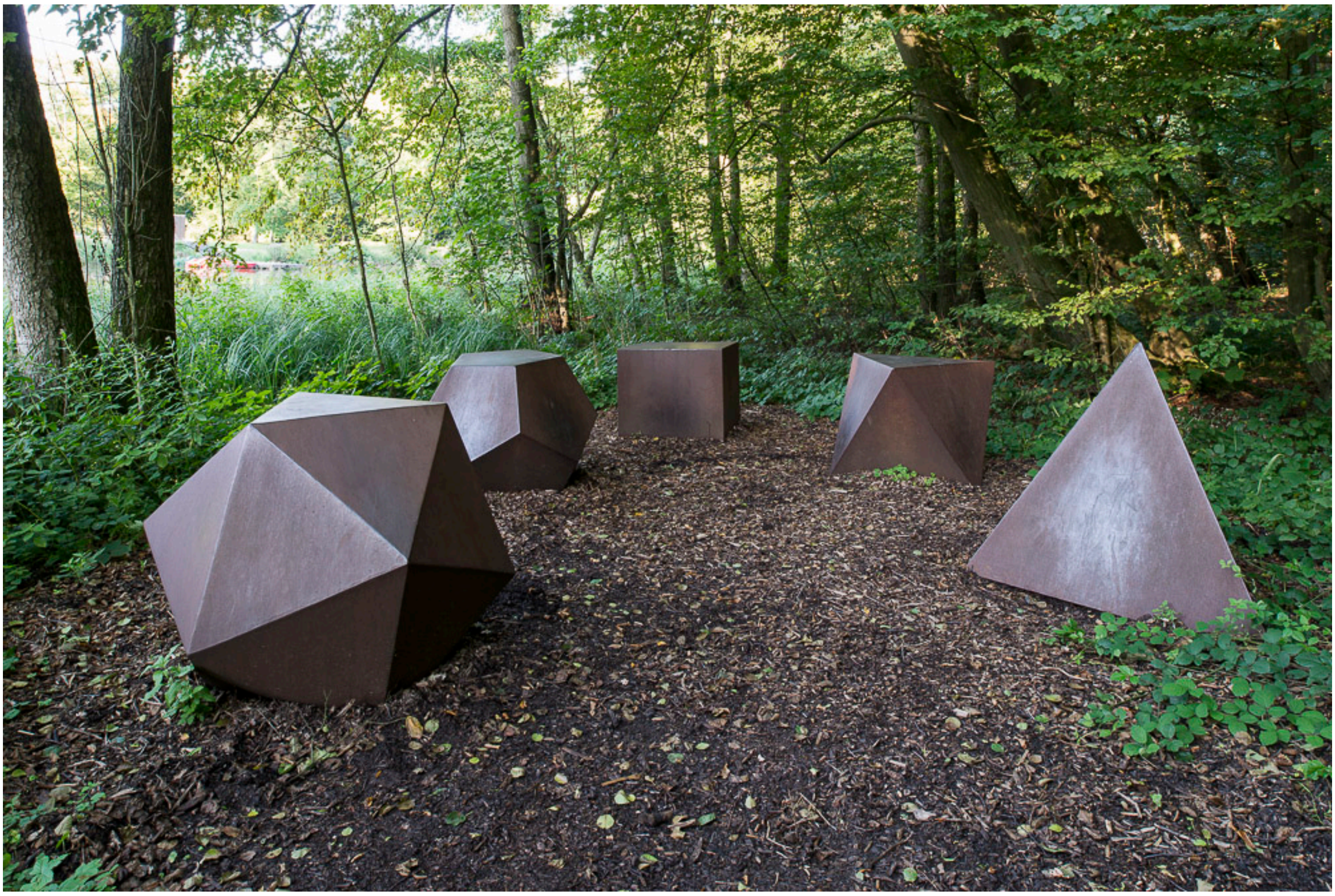
Mathematical Invariants in Nature and the Arts: Platonic Solids

by Pier Francesco Sciuto

Starting with minerals

When visiting a collection of geological specimens, one of the first things that catches the eye is the vivid colors of the minerals. As you get closer, your attention will turn to the geometric shapes of the various crystals, with the richness and variety of possible forms found not only in different substances but also within the same substance. For example, pyrite, a very common mineral, often occurs as cubic crystals, whose crystal lattice is indeed a cube, but it can also be found in crystals that, while based on the same lattice, take the form of octahedrons or other more complex geometric solids: for instance, the dodecahedron, a solid figure with twelve pentagonal faces.

It is also quite common to observe two or more different crystalline forms in a single specimen. From this perspective, one could say that the discovery of simple forms, all traceable to a single polyhedral shape, is the exception rather than the rule. For those who wish to become familiar with geometric solids, art history offers a vast and accessible field of study. In particular, the Platonic solids are an excellent resource for this exploration.

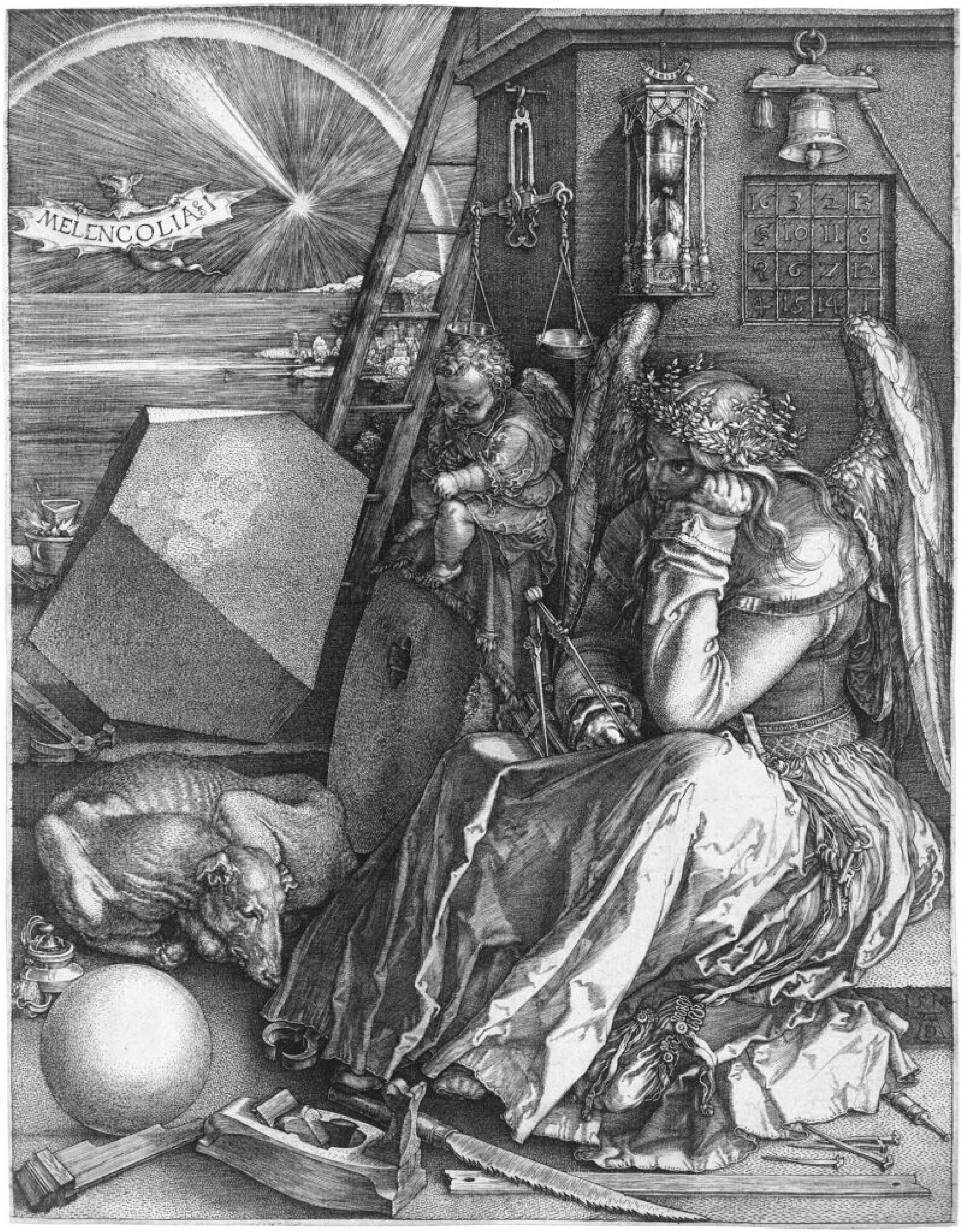


Ekkehard Neumann, The Five Platonic Solids (from left to right: icosahedron, dodecahedron, cube, octahedron, tetrahedron), 1996, steel, each element approximately 1 m high, Steinfurt, Bagno Park (photo credits Thorsten Arendt).

The five polyhedra

We don't know who first undertook the study of Platonic solids. What is certain is that a systematic investigation of polyhedra – that is, solids bounded by a finite number of flat polygonal faces – began with Greek civilization. The first written account comes from Plato (428–347 B.C.). In *Timaeus*, the great philosopher describes five regular convex polyhedra. These are the five solids that would later come to be known as "Platonic solids." They are, in order: the regular tetrahedron, with four equilateral triangular faces; the cube, or hexahedron, with six square faces; the octahedron, composed of eight equilateral triangles; the dodecahedron, bounded by twelve regular pentagons; and the icosahedron, with twenty equilateral triangular faces.

The identification and description of the five regular polyhedra are traditionally attributed to the Pythagorean school (c. 530–450 B.C.). It is well documented that the Pythagoreans established a connection between the regular polyhedra and the four constituent elements of the cosmos: earth, water, fire, and air. These constituent elements would be revisited by many other scholars, including Empedocles of Agrigento (484/481–424/421 B.C.), who regarded them as the “roots” (ριζώματα, rizòmata) of all things. The Platonic solids thus became so important in the Hellenic cultural landscape that they came to be firmly associated with the scientific and epistemological perspectives that were gradually gaining prominence. In them lay the visual, imaginative counterpart to a sort of “theory of everything”. Fire, being lighter, was thought to consist of tetrahedra, while earth, with its stability, was thought to consist of cubes. Water, a flowing element, was associated with the icosahedron, the solid most similar to a sphere and therefore the easiest to roll. Air was enigmatically associated with the octahedron. Finally, the dodecahedron was regarded as a symbol of the universe and of the crystalline element that constituted it: aether. In turn, the Platonic solids became emblems of Hippocrates’ (c. 460–c. 377 B.C.) humoral theory of medicine, in which human beings could be classified according to the four fundamental temperaments (sanguine, melancholic, choleric, phlegmatic). Finally, colors (black, white, yellow, and red) were also incorporated into this multifaceted system of thought.



Albrecht Dürer, Melencolia I, 1514, engraving, 23.9 x 18.9 cm (Wikimedia).

A work of art that, at the height of the Renaissance, embodies and synthesizes classical thought, specifically in the realm of mathematics and geometry, is *Melencolia I*, the famous engraving created in 1514 by Albrecht Dürer. Numerous symbolic elements (a magic square, symbols of labor, measurement, and distance) are brought together here to evoke the emotional state, akin to the inscrutable fervor of creativity and ideas, referred to in the title. Dürer also introduces a mysterious polyhedron, a sort of cube with two truncated vertices. In the engraving, we see a central female figure with a dark face (the nigredo of alchemical doctrine), which can be interpreted in terms of the “melancholic humor” or “sloth.” In the background is a black sun. The central figure is winged, and it is therefore reasonable to assume that she will rise from her current condition toward a state that promises perfection. The same meaning can be attributed to the black sun, which is eclipsed but crowned by a rainbow.

The idea of perfection and purification is also symbolized by the magic square shown in the upper right corner. It is a numerical matrix in which the number 34 symbolizes perfection, as it is the sum of every row, every column, and every diagonal. In magic squares, this sum is called the magic constant. The number is composed of 3 and 4: in alchemy, 3 corresponds to the three principles of the material world (sulfur, salt, mercury), while the number 4 refers to the four-part division of the Great Work, that is, Nature. The tradition of magic squares dates back to around 2000 B.C. and continues to the present day. Their introduction to the Western world dates back to the 14th century, mediated by Arabic sources, which in turn drew upon Indian and Chinese cultures.

SOLID	*ELEMENT*	*TEMPERAMENT*	*COLOR*
HEXAHEDRON (CUBE)	EARTH	MELANCHOLIC	BLACK
ICOSAHEDRON	WATER	PHLEGMATIC	WHITE
OCTAHEDRON	AIR	CHOLERIC	YELLOW
TETRAHEDRON	FIRE	SANGUINE	RED
DODECAHEDRON	UNIVERSE (ETHER, QUINTESSENCE)		

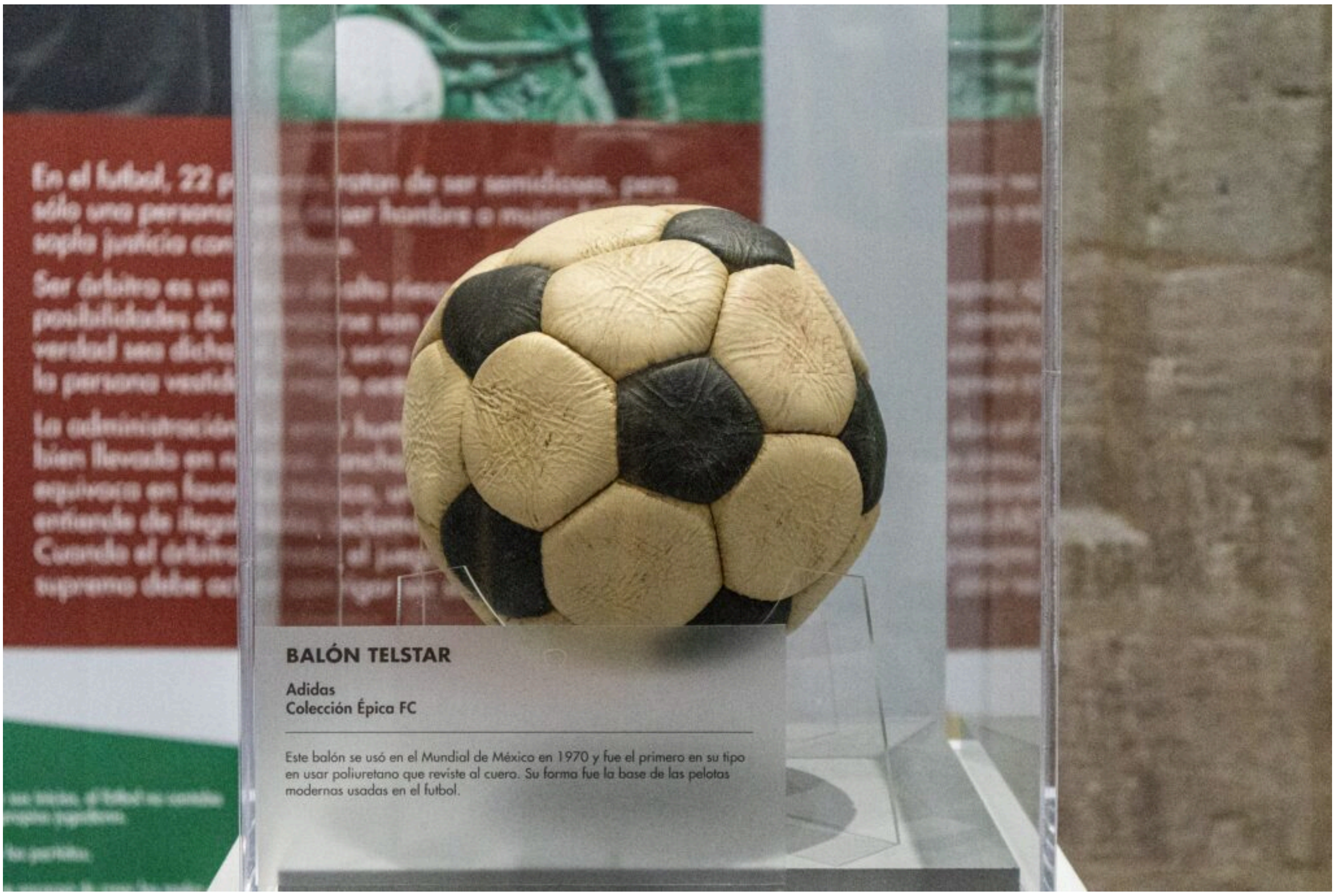
SOLID	DAY	SEASON	AGE	ALCHEMY
HEXAHEDRON (CUBE)	NIGHT	WINTER	OLD AGE	NIGREDO (PUTREFACTION)
ICOSAHEDRON	DAWN	SPRING	CHILDHOOD	ALBEDO (PURIFICATION)
OCTAHEDRON	NOON	SUMMER	YOUTH	CITRINITAS (DISTILLATION)
TETRAHEDRON	SUNSET	FALL	MATURITY	RUBEDO (SUBLIMATION)

The classical and medieval conception of the connections between Platonic solids and: a) the elements, humors, and colors; b) the phases of the day, the seasons, the ages of life, and the alchemical stages.

In the classical and medieval view of the connections between the Platonic solids and the humors, the elements, and the colors, there is a hierarchy of knowledge, whose levels symbolize the four material spheres that the neophyte must traverse to ascend to God. Earth represents the chaos and darkness of the uninitiated; water – that is, baptism – is the emblem of outward regeneration through triumph over temptations; air signifies the divine truth illuminating the neophyte's intellect; fire, or the supreme degree, opens the heart to divine love. Alchemical doctrine explores the Platonic solids, attributing further meanings and nuances to them, drawing precisely from the reading of *Timaeus*, a text widely known through the Latin translations made by Cicero and Calcidius. Such is the importance attributed to this work over the centuries that, in Raphael's fresco *The School of Athens*, the figure of Plato, positioned at the center of the upper section, with Aristotle to his left, is depicted holding the *Timaeus* itself.

Other polyhedra

The sphere features prominently in Pythagorean thought, but no specific meaning is attributed to it. It is, however, studied by Euclid, who devotes the thirteenth and final book of the *Elements* to an exclusively scientific examination of the five Platonic solids, in order to calculate the characteristics of each. Following Euclid, Archimedes (c. 287–212 B.C.) and Apollonius of Perga (262–190 B.C.) also studied the Platonic solids. Thirteen new polyhedra, known as Archimedean or semiregular polyhedra, were thus identified and described. The importance of these polyhedra lies in the fact that they are very common in nature. One example is *Clathrus ruber*, a fungus whose shape corresponds to a truncated icosahedron. But as always, the gap between nature, science, and the collective imagination is narrower than one might think, giving rise to surprising metamorphoses.



A vintage Adidas Telstar soccer ball on display at the exhibition "¿De qué color pinta el verde?", Mexico City, El Colegio Nacional, 2026 (Wikimedia).

In fact, the shape of the classic soccer ball – composed of twelve black pentagonal faces and twenty white hexagonal faces, for a total of ninety edges and sixty vertices – corresponds to a truncated icosahedron. Adidas patented this design, which was used in the 1968 European Championship held in Italy and the 1970 World Cup played in Mexico, and it remains an unsurpassed archetype to this day. It was named Telstar, by analogy with the satellite that, at the time, provided intercontinental television broadcasts. The alternating black and white pattern was intended to ensure better television visibility compared to the classic brown-striped leather ball. About fifteen years later, in 1985, chemists Harold Kroto, Robert Curl, and Richard Smalley developed a compound whose molecules consisted of sixty carbon atoms, bonded to form the C60 molecule, known as fullerene. For this discovery and its technological applications, the three were awarded the Nobel Prize in Chemistry in 1996. It is worth noting that the family of compounds known as fullerenes is named after Richard Buckminster Fuller (1895–1983), the designer and architect who theorized geodesic domes as a futuristic solution for covering and climate-controlling spaces, small, large, and very large, up to the urban scale. In fullerenes, the carbon atoms are arranged at the sixty vertices of a truncated icosahedron; the same structure underlying the Telstar soccer ball.

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Homepage: Raffaello Sanzio, The School of Athens (detail), 1509–1511, fresco, total dimensions 770 x 500 cm, Rome, Vatican City, Vatican Rooms (Wikimedia).
Below: Richard Buckminster Fuller, Geodesic Dome (Spoletosfera), 1967, prefabricated metal elements, Ø 21 m, Spoleto (Manuela Rosi/Wikimedia).